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THE FRACTAL THEORY OF THE SATURN RING

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Abstract

The true reason for partition of the Saturn ring as well as rings of other planets into great many of sub-rings is found. This reason is the theorem of Zelikin-Lokutsievskiy-Hildebrand about fractal structure of solutions to generic piece-wise smooth Hamiltonian systems. The instability of two-dimensional model of rings with continues surface density of particles distribution is proved both for Newtonian and for Boltzmann equations. We do not claim that we have solved the problem of stability of Saturn ring. We rather put questions and suggest some ideas and means for researches.

1 Introduction

Up to the middle of XX century the astronomy went hand in hand with mathematics. So, it is no wonder that such seemingly purely astronomical problems as figures of equilibrium for rotating, gravitating liquids or the stability of Saturn Ring came to the attention of the most part of outstanding mathematicians. It will be suffice to mention Galilei, Huygens, Newton, Laplace, Maclaurin, Maupertuis, Clairaut, d'Alembert, Legendre, Liouville, Jacobi, Riemann, Poincaré, Chebyshev, Lyapunov, Cartan and numerous others (see, for example, [1]). It is pertinent to recall the out of sight discussion between Poincaré and Lyapunov regarding figures of equilibrium for rotating, gravitating liquids. Poincaré, who obtained his results using not so rigorous reasoning and often by simple analogy, wrote, "It is possible to make many objections, but the same rigor as in pure analysis is not demanded in mechanics". Lyapunov had entirely different position: "It is impermissible to use doubtful reasonings when solving a problem in mechanics or physics (it is the same) if it is formulated quite definitely as a mathematical one. In that case, it becomes a problem of pure analysis and must be treated as such". Both great scientists had in a sense good reasons. When it comes to real natural processes, one cannot be aware that all external and interior effects were taken into account. And so, any model is unusable as a formal proof that the real process will behave in one or another way. As for figures of equilibrium for rotating, gravitating liquids, both mathematicians did not take into account its non-homogeneity (in contrast to Clairaut), neglected of interior currents (in contrast to Riemann and Dedekind), ignore electro-magnetic phenomena. No one, so far as we know, tries to consider this last effect on figures of equilibrium, though no doubt a close relations of electro-magnetic phenomena with the gravitation must exist. It is suffice to mention the electro-magnetic dynamo of Earth core. So, the refusal of Poincaré to perform proofs with scrupulous attention is psychologically justified. But on the other hand, the point of view of Lyapunov, being more difficult, is more attractive for mathematicians, because the essence of mathematics is the abstract. If only one has an intuitive assurance that the main cause is found, the purely formal proof is very much desirable. As a matter of fact, the mathematical rigor is intimately associated with the mathematical beauty.

Let us recall main facts about the history of explorations of the Saturn Ring. When Galilei directed his new made primitive telescope to the heaven, he discovered, apart from the Jupiter satellites, something strange related with Saturn. He could not understand whether it was a triple joint ball or a ball with handles. Hence, he encode his discovery by the famous anagram: "Altissimum planetam tergeminum observavi" which means: "I have observed the very remote Planet being triple". A diversity of conjectures had been put forward about this phenomenon. Galilei himself considered Saturn as three different planets situated on the same ray of sight or as the planet with two satellites and sometimes called it "ears". When the Ring appears to vanish, turning edgewise, he wondered: if Saturn as in the myth really had swallowed his children. And only the great disciple of Galilei — Christian Huygens — possessed courage to believe in his own eyes and to behold something incredible: the ring of regular shape hovering weightlessly in vacuity around the planet! Recently Rings (though not so big as the Saturn Ring) were discovered around all planets besides that of Earth-group: Jupiter, Uranus and Neptune have Rings.

The first gap (the Cassini gap) of distribution of mass density in the Ring was discovered in 1675. It has the width about 4800 kilometers. Later on a number of gaps with the width of hundreds kilometers (Enke gap, Laplace gap, Huygens gap among others) were detected. Situation was fundamentally changed after flights of cosmic spacecrafts: Voyager 1 and 2, Pioneer, and NASA's Cassini expedition. Photos that was made from these apparatuses allow to calculate (it is difficult to believe!) until cent hundreds concentric ringlets and the corresponding numbers of gaps and divisions. The exact number is unknown because some ringlets stick together and even intertwine to one another.

The theoretic investigations of the Saturn Ring began immediately after its discovery. Researchers tried to find conditions that enable the stability of the Ring. The stability as itself was of no doubt being evident result of continues observations

But after discovery of the thin structure of the Ring, efforts of scientists switch into explanations of the necessity of the dividing the Ring on big number of ringlets. We will not speak on groundless fantasies similar to the p-adic analysis, to the quantization of orbits, to black matter and so on.

The point of view of this work is as follows: the exact solution in the ideal case of the flat Ring that consists of infinitesimal particles has to contain infinitely many ringlets. This thesis is based on our theorem on generic fractal structure of trajectories of piece-wise smooth Hamiltonian systems. [27].

There exists traditional distinction between three main conjectures: the Ring is entirely rigid, or it is liquid or it consists on many distinct particles (so called Cassini conjecture).

Conjecture of rigid Ring was disproved by Laplace [13]. He proved that in case of constant density the rigid Ring would be unstable. The arbitrary displacement of its center relative to the gravity center of Saturn must lead to increasing of the anomaly, and the Ring will collapse on Saturn. Hence such Ring has to be inhomogeneous.

The remarkable work of Maxwell [15], that was honored by Adams prize, anticipated all subsequent researches of the Saturn Ring on one and a half centuries. At the beginning of his paper Maxwell apologized to readers that he had explored the question, which is not bounded with any practical benefit neither in navigation nor in astronomy, being moved

by scientific curiosity only. The method of Maxwell consists in consideration of small oscillations relative to the equilibrium state as solutions of variational equations. Nowadays, as a result of progress in creation of modern telescopes and especially due to flights of cosmic apparatuses to Saturn, oscillating regimes became directly observable. One can see spiral changes in density, bends of the disk, spokes, and so on. In particular, Maxwell considered the question about the *nonhomogeneous* rigid Ring. Under the supposition that the homogeneity is violated in a unique point where exists an additional mass, he proved that it will be necessary for the stability that this mass consists as a minimum $4/5$ of the total mass of the Ring. But it means that actually it will be a satellite with a comparatively small additional tail of asteroids, that contradicts the observations.

Rado explored the case of continuous nonhomogeneous distribution of masses. He proved that the necessary conditions for stability is the following: the variation of the mass density has to be changed from 0.04 to 2.7, that is no less than in hundred times that again contradicts the observations. The Ring is extremely relatively thin, and if some of its part was more thin that an other, than (be Girn calculations) to sustain the satellites attraction, its rigidity must be 1000 times as great as that of the steel.

So the Ring cannot be rigid.

To explore the impact of the Ring itself on the behavior of its particles Poincaré [17] had found the main part of asymptotic of the integral that describes the force of attraction that acts on a point P from a circle with the uniform linear density. He introduced nice, new construction: the arithmetic- geometrical mean. He proved that the main part of the asymptotic of the attraction force for P lying sufficiently near to the circle is the arithmetic-geometrical mean of the nearest and the farthest distant from P to points of the circle. Poincaré used this result in investigation of the model of the tore-like liquid Ring with the ellipsoidal meridional section. Poincaré proved its instability.

Maxwell [15] [16] proved that the ultimate density of the Ring which is compatible with the stability (for the *liquid* Ring as well as for the dusty ring) does not exceed $1/300$ of the density of Saturn. It is called the Maxwell limit. Poincaré showed that if the *liquid* Ring rotates with the speed of satellite of the same orbit than its density must be greater than $1/16$ of the planet density. It would be incompatible with the Maxwell limit. So, the Ring cannot be liquid.

We are forced to accept Cassini conjecture: The Ring consists of the set of distinct rigid asteroids. It counts in favor of this conjecture that the Ring appears transparent; stars and the Saturn surface are seen through it and the light passes the Ring without refraction. The term itself "the Cassini conjecture" should be considered as anachronism. Now it is not a conjecture but the universally accepted model that is verified by repeated well established facts of observational astronomy.

I remember a fantastic impression when I observes for the first time the Saturn Ring through a telescope. It was striking to see such a strange, excellent, and regular form among stars. At first, let us try to give a very rough, heuristic explanation of the phenomenon.

Saturn, having very big mass, attracts particles from the ambient space. Consider an auto-gravitating cloud of particles subject to mutual collisions in the field of gravity of

Saturn. If we mentally switch out the gravitational force of Saturn and the process of colliding, then we obtain a flat disc $R < \bar{R}$ as a stable equilibrium [1]. Now let us switch on the gravitational force of Saturn and the process of colliding. Let $\rho(R)$ be the density of particles, $l(R)$ be the length of free run.

Claim 1. .

There exist positive constants ρ^* , ε , and R^* such that the density of particles generate an annulus

$$\rho(R) = \begin{cases} \varepsilon & \text{for } R < R^* \\ \rho^* - \varepsilon & \text{for } \bar{R} < R < R^* \end{cases} \tag{1}$$

Indeed. The probability of collision is growing together with the density. Hence, in the domain $R < R^*$, where the density is very low, the length of free run is big and collisions are scarce. The stochastic change of the angular velocity is significant and the corresponding particles with the big probability fall into Saturn. So, the density of particles in the vicinity of Saturn has the tendency to fall. On the contrary, in the domain $\bar{R} > R > R^*$, where the density of particles is high, collisions happen frequently. As a result the angular velocity attains its stationary value. The density changes negligible. In addition, the stock of particles is supplemented from the ambient space due to gravitational force of Saturn.

So, in our rough approximation we obtain the step-function for the density of particles that generates the Ring of Saturn.

Let us remark that this argumentation relates not only to the Saturn Ring but to all big planet. Now it is known that Jupiter, Saturn, Uranus, and Neptune have rings of its own.

In the present work it will be shown that the mass distribution in the flat Ring consisted of the set of isolated gravitated particles must be very irregular. It cannot be given by smooth positive function. The statement is in accordant with the observed decomposition of the Ring on numerous very small ringlets.

2 The model of flat Ring (Model A)

Let us give some facts for convenience of readers. Particles of the Ring consist in general of snow and ice with minor amount of carbon and silicon. The size of particles varies from microns to meters (very rare till kilometer). Diameter of the Ring is approximately 480 000 km. Its thickness from 1 to 1.5 km. The mass of Saturn is $\mathcal{M}_{\mathcal{S}} \approx 6 \cdot 10^{29} g$. The mass of the Ring is $\mathcal{M}_{\mathcal{R}} \approx 3 \cdot 10^{22} g$. The mass of the Mimas (one of the nearest Moon of Saturn) is $\mathcal{M}_{\mathcal{M}} \approx 3 \cdot 10^{19} g$.

The thickness of the Ring is insignificant in comparison with its size. Hence, the approximation of the flat Ring seems reasonable. Let us give the following reasoning as an additional argument. Take the carrying plane of the Ring as the plane Ox_1x_2 and its normal in the center of the Ring take as the axis Ox_3 . Particles, that for some reason appears out of the plane Ox_1x_2 , are subject of action of the additional attractive force

from the Ring. If the particle appears sufficiently far from the edge of the Ring end sufficiently near to its plane than the vertical component of the attraction force induced by the Ring is practically constant. It is approximately equal to the force of attraction of infinite plane Ox_1x_2. Hence, particles leaving the plane Ox_1x_2 in addition to the movement under the action of the central force execute harmonic oscillations directed by the axis Ox_3. After averaging over these frequencies, the dependence on x_3 disappears and it remains only along the plane Ox_1x_2. This heuristic argumentation shows that two-dimensional model has to be adequate to the situation.

So, at first we consider a collision-free two-dimensional model (call it Model A) for particles lying in the round annulus Q with the centrum at the point O. The pare of cartesian coordinates (x_1, x_2) will be denoted in what follows by a single capital letter X. In the plane Ox_1x_2 we introduce polar coordinates R, ϕ in addition to the cartesian coordinates. Let the internal board of the Ring be $R = R_1$; the external one be $R = R_2$.

Assumption 1. *Suppose that a closed, two-dimensional annulus Q is filled by particles with the surface density $f^0(R)$ that is symmetric relative to the group of rotations $\mathbb{O}_2$. Let the support of the function f_0 be Q. Suppose that the density is continues and positive at points of Q. It follows from the compactness of Q that*

$$f^0(X) > \alpha > 0 \tag{2}$$

for $X \in Q$.

Let us find the main member of the asymptotic of attraction forces of the Ring which act on points lying near its boarder.

Consider a domain $\Omega \subset \mathbb{R}^2$ with smooth boundary and with the density $f(X)$ of mass distribution. The resultant force of attraction from the mass which contained in Ω acting on a sampling point X will be denoted by $F(X)$. The constant of gravitation we take as 1.

Claim 2. *Let the function $f(X)$ in a neighborhood of a point $X_0 \in \partial\Omega$ meets conditions of the assumption 1. Than the main part of the asymptotic of attraction forces $F(X)$ as $X \to X_0 \in \partial\Omega$ is equivalent to $-C \ln |X - X_0|$.*

Proof.

Let the point X_0 be O, and consider the domain Ω_ϵ with the mass density $f(x)$, defined by inequalities $\Omega_\epsilon = \{-A \le x_1 \le A,\ \epsilon \le x_2 \le B\}$. The sign of ϵ is arbitrary. Let us estimate the main part of the attraction force acting on a point O from the domain Ω_ϵ. In view of (2), one has

$$\int_\epsilon^B dx_2 \int_{-A}^A \frac{x_2 f(x_1,x_2)}{(x_1^2+x_2^2)^{3/2}} dx > 2\alpha \int_0^A \left\{ \frac{1}{(x_1^2+\epsilon^2)^{1/2}} - \frac{1}{(x_1^2+B^2)^{1/2}} \right\} dx_1 \approx C \ln \frac{1}{|\epsilon|}. \tag{3}$$

□

Let us denote by $\mathcal{F}(R)$ the lamp gravitational force acting on a particle from Saturn and its Ring. Equations of movement in the model A are

$$\left\{ \begin{array}{c} \dot{X} = V \\ \dot{V} = \mathcal{F}(R) \end{array} \right. \tag{4}$$

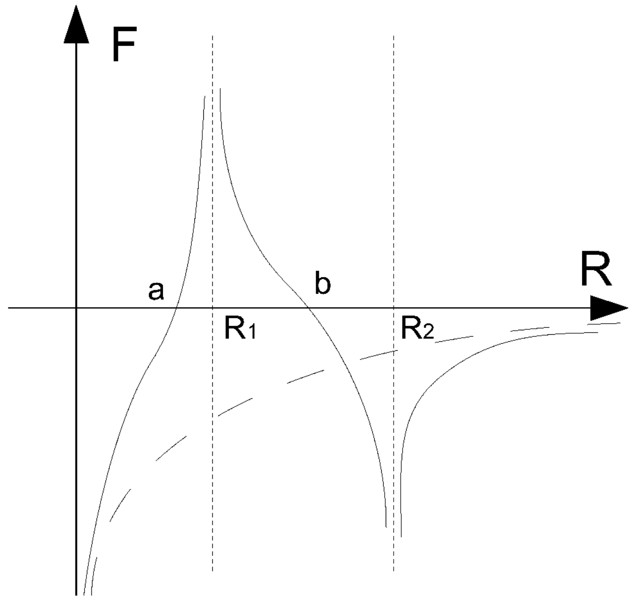

Figure 1: Attracting forces of the Ring and Saturn

Theorem 1. *Let the assumption 1 be fulfilled. Than the state that is defined by the function $f^0(R)$ is unstable.*

Proof.

The additional force of gravitation, that was induced by the Ring itself, acting at points lying far from the edge of Ring, is small comparatively to that acting from Saturn. The reason is that the density of the Ring does not exceed $1/300$ of the density of Saturn. Besides, for points P that is placed inside of the Ring, the force from the inner (relative to P) part of the Ring is almost balanced by the outer part. Meantime, the formula (3) follows that the Ring attraction infinitely grows as P tends to the boarder of the Ring.

Due to the $\mathbb{O}_2$-symmetry, it is necessary for $f^0(R)$ to be steady-state that the function F(R) does not change along trajectories. Hence, particles have to move along circles. But for particles with $X_0 \in \partial Q$ this movement is demanded the infinite speed that is physically non-realizable. So, in the real movement, particles would move inside the Ring (both for points lying at $R = R_1$ and that lying at $R = R_2$). Hence if the Ring appears in the state that meets the assumption 1, than particles of the outer part must tend to approach Saturn, while that of the inner part to withdraw from Saturn. As a result the Ring as a whole should begin to contract. This process will proceed till the approximation of the flat Ring becomes inadequate. Therefore the assumption 1 leads to instability.

□

Though it is not directly connected with the proof of the theorem, it will be interesting to follow the evolution of the system with the distribution $f^0(t_0, R)$. Taking into account that the total force from Saturn and from the Ring is directed by the radius, one can draw the graph of the force $\mathcal{F}(R)$, that is shown schematically on the Fig 1 (the exact form of the graph depends on the distribution function f).

In the simplest case when the distribution function is constant on the interval $[R_1, R_2]$, the function $\mathcal{F}(R)$ monotonically increases from $-\infty$ to $+\infty$ on the interval $(0, R_1)$; monotonically decreases from $+\infty$ to $-\infty$ on the interval (R_1, R_2) and monotonically increases from $-\infty$ to zero on the interval $(R_2, +\infty)$.

In this case there exist exactly two points of intersection of the graph of the function $\mathcal{F}(R)$ with the absciss axis: $R = a$ and $R = b$. These points correspond to libration lines. The name libration means that these lines consist of fixed points of the system (4). It follows from consideration of signs of the function $\mathcal{F}(R)$ that the circle $R = b$ consists of stable equilibrium while the circle $R = a$ consists of unstable ones. By using the theorem about continues dependance on initial data for the system (4), one can prove that the density f remains to meet conditions of assumption 1 in the process of growing of t. Indeed, the circular symmetry will be conserved and the strictly positivity on the interval $[R_1(t), R_2(t)]$ follows from the matching of the density in corresponding under the transference points and from the theorem about the integral invariant for area. Hence, the function $R_1(t)$ monotonically grows and the function $R_2(t)$ monotonically decreases. The Ring will contract to the direction of the circle $R = b$. But we have to take into account that the value b itself will change in this process.

One can put in doubt the existence of nonzero limit of the density at the border of the Ring. But the observations show, that the Ring and each of his ringlets have the precise, accurate boundaries, instead of fuzzy boarder.

Corollary 1. *The theorem 1 remains valid for any sufficiently wide sub-ring which is divided from other parts of the Ring by two divisions that is free from gravitating particles.*

In the model A we did not take into account effect of collision of particles which a priory can substantially change the situation. We have to consider the kinetic Boltzmann equation. Consider the corresponding model B when effect of collisions is commensurable with the dynamical ones.

We have to modify a little the Boltzmann equation to apply it to the two-dimensional situation, to the rotation of particles in the Ring, to take into account the distribution of mass of particles, and lows of its scattering.

3 Kinetic Boltzmann equation (model B)

To explore the stability of the Ring with regard to collisions we have to apply methods of statistical physics and ergodic theory [6], [9]. Consider the integro-differential kinetic Boltzmann equation [2], [22]. In 1932 Carleman [3] proved the first global existence theorem for Boltzmann equation in the space-homogeneous case. In 1963 Grad [7] proved the general local existence theorem in the neighborhood of the Maxwell distribution. Later on this theorem was repeatedly generalized (see, for example, the bibliography in [23]).

The Boltzmann equation in connection with the Saturn Ring was considered in works of Friedman and Gorkavyj [8]. Their predecessors sought reasons of stability of the Ring because it does not destruct almost 4 century. But Friedman and Gorkavyj began to seek reasons of instability, namely, to seek oscillation processes making the Ring to stratify in big number of thin ringlets while preserving it as a whole. They assume that stratification of the Ring is summoned by non-equilibrium wave-like steady processes which take place in the Ring.

The kinetic Boltzmann equation plays the leading part in the thermodynamics, the hydrodynamics, and the quantum mechanics. In particular, from this equation follow

Navier-Stokes equation, Euler equation, Vlasov equation for plasma... (see, for example, [9], [19], [23]).

To use Boltzmann equation one needs that the time of free running essentially exceeds the time of colliding. This condition is fulfilled in our situation. Indeed, mutual relative speeds between particles of the Ring are estimated by values of several cm/cek. The Ring is transparent, that means that sufficiently amount of rays of the light do not meet particles of matter on his way through the Ring, i.e. on a distant about 1-1.5 km. Besides, estimations of the density and of mean size of particles show that the time of free running are estimated by several hours. So, the main conjecture about the time of collisions which is the base of derivation of the Boltzmann equation may be thought of as fulfilled.

Let us denote by $f_\mu(t, X, V)$ the scalar density of distribution of particles with the mass μ in a moment t at a point $X \in Q$ that have the velocity V. Scalar product of functions from the velocity V will be defined by the metric L_2 and will be designed by angular brackets. For instance, the mean value of the function $\phi(V)$ for the density $f(V)$ equals

$$< \phi(V), f(V) >= \int \phi(V) f(V) dV.$$

Let us introduce the main macroscopic values, defined by a distribution f. We use here the standard terminology of the kinetic theory of gases. Note that the term "microscopic value", being applied to particles with size around a meter (and even reach amount a kilometre), seems at any rate strange. But one cannot forget about astronomical size of the Ring itself.

The macroscopic density of particles equals

$$\rho(t, X) =< 1, f >= \int f(t, X, V) dV. \tag{5}$$

The mathematical expectation of velocity per unit of mass is defined by the vector U with coordinates

$$u_j(t, X) = \frac{1}{\rho(t, X)} \int v_j f(t, X, V) dV. \tag{6}$$

Denote by $W(t, X) = V - U(t, X)$ the chaotic velocity of particle (the deviation of the real velocity from its mathematical expectation).

Kinetic temperature per unit of mass equals

$$T(t, X) = \frac{1}{2\rho(t, X)\kappa_B} \int W^2(t, X) f(t, X, V) dV, \tag{7}$$

where κ_B is the Boltzmann constant. The term kinetic temperature does not imply neither the temperature of ambient space nor that of inside of particles. The kinetic temperature is defined (as in kinetic theory of gases) as dispersion of a random field of velocities of particles contained in the Ring.

The kinetic pressure is defined by the formula

$$p = \kappa_B \rho T,$$

The last relation is called by the equation of states in kinetic theory of gases.

Unlike of the classical deduction of the Boltzmann equation we have to take into account that particles can have all a priory admissible masses. Hence one has to consider a stochastic distribution $s(m)$ of masses. As a pattern we take the Boltzmann arguments [2] who considers the mixture of two gases: one consists of molecules of the mass m_1 and other of the mass m_2. Since the Boltzmann reasoning relates to the single act of collision, it can be applied to the mixture of gases with any given mass distribution, by introducing continuum distribution functions to suit the mass of particles.

Given the equations of particles dynamics (4) and the rules for the changes its velocities after collisions one can write out the equation for change of the density.

So, the kinetic Boltzmann equation is the balance equation for change of the particles density of various masses subject to forces $\mathcal{F}(X)$ and subject to changes of the movement due to collisions.

$$\frac{\partial f_\mu(t,X,V)}{\partial t} + \sum_{i=1}^{2}\left(\frac{\partial f_\mu(t,X,V)}{\partial x_i} v_i\right) + \sum_{i=1}^{2}\left(\frac{\partial f_\mu(t,X,V)}{\partial v_i}\mathcal{F}_i(X)\right) = J[f], \qquad (8)$$

where $J[f]$, that is called the collisions integral, is a functional from the distribution function f having the form

$$J[f] = \frac{1}{\lambda}\int_0^\infty s(m)dm \int K(\sigma)[\tilde{f}_\mu \tilde{f}_m - f_\mu f_m] d\omega_\mu d\sigma = 0. \qquad (9)$$

Here $d\omega_\mu$ is the volume element at the point, where the particle of the mass μ lays; the kernel of the integral $K(\sigma)$ describes the result of collision of two particles; the value λ is the time of free run. The sign tilde over letters (say, $\tilde{f}$) shows on values of distribution function at points that relate to the velocities after collision.

As it is known, the unique stationary (does not depend on time) stable solution to the Boltzmann equation in d-dimensional space is obtained only when the particles velocity have the Maxwell distribution.

$$M_{(\rho,U,T)}(V) = \frac{\rho}{(2\pi\kappa_B T)^{d/2}} \exp\left(-\frac{|V-U|^2}{2\kappa_B T}\right). \qquad (10)$$

In the two-dimensional case the exponent $d/2$ that stands into denominator equals 1. The particles density distribution relative to the masses is a result of subdivisions and sticking together of particles in the colliding process. Parameters characterizes collisions depend on relative velocities of colliding particles. So, it is natural to suppose that the distribution of mass particles in the process of relaxation of equilibrium with the passage of time get adjusted to the velocity distribution and become Maxwell-like.

The case is reduced to the addition into the Boltzmann equation one more variable related to mass and to the introduction the set of distribution functions which relate to each value of masses. One has only to add under the integral sign the same factor as that for the Maxwell distribution relative to velocities. Hence, integrals of collisions remain in practice the same Gaussian integrals as for the one-particle equation.

Collision integrals for the Maxwell distribution M vanish and the system (8) that consists of continuum equations decomposes into independent of one another scalar equations.

Methods of exploration of non-stationary solutions to the Boltzmann equation were developed in [4], [3], [7], [10], [23], [14] a.o.. The basic method is to expand the unknown distribution function in a series (as in perturbation theory) by moments of some fixed balanced distribution f, as a rule, of Maxwell distribution M. The benefit of such approach is that the calculation of moments needs only the finite number of the expansion coefficients (its number is equal to the degree of the moment).

Assumption 2. *Suppose that forces $\mathcal{F}$ do not depend on time, and depend only on the distance R from the origin, and are defined by potential of the field of Saturn and the Ring. Suppose that the space distribution of particles is given by a smooth, positive density. Suppose that the speed distribution $f_\mu^0(\cdot,\cdot,V)$ and the corresponded mass distribution both are locally Maxwell-like.*

The term "locally Maxwell" in this situation means that parameters of Maxwell distribution (the mean value and the dispersion) depend only on the distance from the origin.

It was shown above that if one does not take into account collisions than the corresponding movement is unstable. Now we are in the frame of the thermodynamics and the kinetic theory when the movement is described by the Boltzmann equation (8). This model can be regarded as an analog of the shear two-dimensional Couette flow [5], [14], [20], [21], when parallel walls that carry away the fluid have constant but differing velocities. In that case the fluid is stratified on fibers with constant velocities that vary linearly from one fiber to another. The difference consists in the following: in our case the sheared in the radius rotation is caused by the external forces of gravitation that depend on the radius. To describe this situation it is used the term differential rotation.

So, we have two-dimensional model (call it as model B) of the

Ring Q with the differential (depending on R) rotation being induced by the gravitational forces of Saturn and the Ring.

Some theoretic arguments and experimental photometric data show that ice particles of the Ring are covered by a thin loose sediment that is made of snow. This layer soften strokes of particles that take up a part of kinetic energy. This process leads to equalization chaotic velocities. That means decreasing of the dispersion i.e. the lowering of the kinetic temperature. But there exists an opposite tendency that prevents cooling the Ring: it is heating due to sheared viscosity, i.e. from the friction between fibers with different speeds of rotation (as in the Couetta flow). At the state of equilibrium shear heating must compensate the dissipation of the energy in collisions.

Theorem 2. *The stationary steady state solution to the system (8) of the Boltzmann equation for the model B with free boundary conditions cannot be given by distribution functions $f_\mu^0(X,V)$ which meet assumptions 1 and 2.*

Let us recall that these assumptions mean that functions $f_\mu^0(X,V)$ are C^1-smooth and bounded from below by a positive constant in the domain Q. Secondly they have local Maxwell distribution with parameter depending only on R, and the same is true for the mass distribution.

Proof.

We do not need in the complete using of methods of Chapman and Enskog [4] and to calculate the exact steady state distribution. It will be suffice to show that such distribution cannot meet conditions of the assumptions 1 and 2.

To describe the transport equation for macroscopic values $\rho(X,t), U(X,t), T(X,t)$ one has to multiple the Boltzmann equation (8) by $1, V, T$ correspondingly and to integrate over V.

In the first case the result of this operation for the density of particles gives

$$\frac{\partial \rho(t,X)}{\partial t} + \int \left(\sum_{i=1}^{2} \frac{\partial f_\mu^0(X,V)}{\partial x_i} v_i \right) dV + \int \sum_{i=1}^{2} \left(\frac{\partial f_\mu^0(X,V)}{\partial v_i} \mathcal{F}_i(X) \right) dV = 0.$$

The right hand site is zero since $f_\mu^0(X,V)$ is the Maxwell distribution. By the same reason, the last integral in the left hand site equals zero. Indeed, after carrying out of the sign of the integral the factor $\mathcal{F}_i(X)$ which does not depend on V, it remains the integral from the full derivative of Gaussian distribution that equals zero.

The integration by parts of the remaining integral gives equation of continuity

$$\frac{\partial \rho(t,X)}{\partial t} + \mathrm{div}(\rho(t,X)U(t,X)) = 0. \tag{11}$$

In the second case one obtains equation for mathematical expectation of velocity (6)

$$\frac{\partial(\rho(t,X)u_j(t,X))}{\partial t} + \int \sum_{i=1}^{2} v_j v_i \frac{\partial f_\mu^0(X,V)}{\partial x_i} dV + \int \sum_{i=1}^{2} v_j \mathcal{F}_i(X) \frac{\partial f_\mu^0(X,V)}{\partial v_i} dV = 0.$$

Again, after carrying out of the sign of the integral factors which do not depend on V and after integrating by parts, one obtains

$$\frac{\partial(\rho(t,X)u_j(t,X))}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left\langle v_i v_j, f_\mu^0(X,V) \right\rangle - \rho(t,X)\mathcal{F}_j(X) = 0.$$

The second moment of the distribution function per the unit of mass is given by the expression $\frac{1}{\rho} P_{ij}(t,X) = \left\langle v_i v_j, f_\mu^0(X,V) \right\rangle$. This expression defines the tensor of pressure $P_{ij}(t,X)$ or, by the other words, the tensor of viscose tensions (Using one or another term depends on what kind of limiting transfer was targeted for consideration). This tensor compensates the lowering of the temperature due to collisions at the cost of shear friction. The preceding equation takes form

$$\frac{\partial(\rho(t,X)u_j(t,X))}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} (\rho(t,X)P_{ij}(t,X)) + \rho(t,X)\mathcal{F}_j(X) = 0. \tag{12}$$

By multiplication of (8) by $(W)^2$ and by integration over V one obtains the equation for the kinetic temperature (7)

$$\begin{gathered} 2\tfrac{\partial(\rho(t,X)\kappa_B T(t,X))}{\partial t} + \int (V - U(t,X))^2 \textstyle\sum_{i=1}^{2} \frac{\partial f^0}{\partial x_i} v_i dV + \\ \int (V - U(t,X))^2 \textstyle\sum_{i=1}^{2} \frac{\partial f^0}{\partial v_i} \mathcal{F}_i(X) dV = -\zeta T(t,X), \end{gathered} \tag{13}$$

where

$$\zeta = -\frac{1}{2\rho\kappa_B T} \int dV (V - U(t,x))^2 J[f]$$

is the rate of cooling induced by collisions of particles. Let us introduce the notation for the vector of the heat flow, that corresponds to the second summand in (13): $q(t,X) = \int(V - U(t,X))^2 f^0(t,X,V)VdV$. Collision integral equals zero. Let us carry out of the sign of the integral factor $\mathcal{F}_i(X)$ in the equation (13) and integrate by parts. The outside of the integral term is zero because f^0 relative to V is Gaussian and the remaining integral equals zero since it gives the mean value of the chaotic velocity (the chaotic velocity itself but not its square). The equation (13) takes the form

$$2\frac{\partial(\rho(t,X)\kappa_B T(t,X))}{\partial t} + \frac{\partial}{\partial x_i}[\rho(t,X)\kappa_B q_i(t,X)] = -\zeta T(t,X). \tag{14}$$

The equation (14) gives something like equation of continuity for the balance of the heat.

Introduce the set of subregions $\Omega_\delta \subset \Omega$ with boundaries $\Gamma_\delta = \{R = R_1 + \delta\} \cap \{R = R_2 - \delta\}$.

The last summand in (12) tends to infinity as $R \to R_1$ and as $R \to R_2$, since the function $\rho(t,X)$ is bounded from below by a positive constant. Consequently, at the steady state, when $U_j(t,X)$ does not depend on t, we have

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\rho(X)P_{ij}(X)) \to \infty \tag{15}$$

as $\delta \to 0$ for any indices j.

Integrate the relation (15) over any subregion $\Lambda_\delta \subset \Omega$ which contains the part Γ_δ of the boundary of Ω_δ for any fixed value of the index j. In the stationary situation the flux of the vector field P_{ij} through Γ_δ directed by the interior normal tends to $+\infty$ as $\delta \to 0$. This flux define forces induced by the pressure P_{ij} that acts on particles of the Ring. The compensation of the pressure would require infinite velocities that is physically impossible, to say nothing of the observed data. The resultant forces at the vicinity of the boundary of the Ring are directed inward the Ring. Particles of Γ must to fly out loosing the gravitational connection with Saturn or to strive toward the interior of the Ring. As a result, the density of particles on a vicinity of the border will rapidly decrease. Consequently, as in the case when collisions do not take into account, under the influence of forces of the pressure, the Ring began contract to its middle part. The state of the disc cannot remain stationary.

Hence the steady state surface density in two-dimensional model B cannot be smooth relative to the space variables.

$\square$

Corollary 2. *The theorem 2 remains valid for any sufficiently wide sub-ring which is separated from other parts of the Ring by two divisions deprived of gravitating particles.*

Theorems 1 and 2 are theorems about non-existence of the expected standard solution. This fact follow that to find smooth positive function that gives steady state stable solution of models A and B is impossible. By the other word these solutions must be unexpected and nonstandard.

We arrive to the necessity to consider irregular distribution functions.

4 The theory of resonances and the concept of fractal behavior

The idea of gravitational resonance is due to Maxwell. He study oscillations of particles of the Ring and distinguished the proper oscillations of the system and the forced ones, evoked by external effects. He did not correlate the external forces with concrete planets or satellites of Saturn. Maxwell considered resonances as phenomena of prime importance. Later on it was discovered one of such resonances: it takes 22.6 hours to the external part of the sub-ring B to make a complete turn, that exactly coincides with the period of rotation of Mimas — the Moon nearest to the Ring of Saturn. Moreover, the farthest point of the external part of this sub-ring (his apo-centrum) always consists the angle $\pi/4$ with the direction to Mimas. After this discovery, astronomers tried to bound any sufficiently big gap with his own “shepherd”, i.e. with a satellite having the commensurable with the gap period of rotation(to cause a resonance).

The term shepherd is a beautiful and bright poetical pattern. It presuppose that each big satellite takes care of his own flock of meteorites that constitute an isolated sub-ring. Admittedly, this approach allows to predict positions of some small satellites of Uranus by using irregularities of gaps in the Uranian Ring. The discovery of the corresponding satellites of Uranus was the real triumph of the concept of shepherds. But as regard to a real physical sense of the term, it seems to us not fully adequate. It is evident that each big satellite affects not solely on a one sub-ring. All the set of satellites (the polyhedron of satellites) affects on each sub-ring. All this structure as a whole is responsible for all microstructure of the Ring.

Friedman and Gorkavyj [8] explored the Boltzmann equation for the Ring. They took as a pattern the method of expanding of solution in moments of velocity distribution of particles relative to the stationary Maxwell distribution. This method was used by Chapman end Enskog for investigation of plasma equations. In the work [8] It was written out the kinetic Boltzmann equation for three-dimensional model of the Ring, consisted of rigid nonelastic particles executed the differential rotation in the gravitational field of Saturn. For simplification of explorations, it was considered a flat (two-dimensional) model which was an analog to the model B and the transport equations for it was written.

Friedman and Gorkavyj certify, that solutions for these equations are unknown. So, they consider the linearization of the equations in the vicinity of Maxwell distribution. It is called by dispersing equation. Further they consider various types of oscillations of the disc density that meet this differential equation. The distinct types of oscillations were obtained after the neglect of one or another member of the equation with the corresponded physical justifications.

The substantial interest represent oscillations of accretion type bounded with the accretion of particles on the Ring from the ambient space. This process supplements the diminishing of particles due to falling some of them on Saturn. Indeed, as a result of collision may appears the loss in the particle velocity and such particles will try to fall on Saturn. But it should be taken into account the following: if one accepts our conjecture of the fractal structure of the Ring, than very thin and extremely intensive jets of meteorites would prevent the falling of particles. Many particles intended to fall on Saturn, being the subject to impulses from jets, get trapped and tangled into the Ring structure.

Conditions for instability for each of selected types of oscillations were found in the work [8] . The necessity of stratification of the Ring on the variety of thin ringlets was explained by these instabilities. What kind of these conditions appears fulfilled in the real situation remains unknown. It is not so bad. It may be that such oscillations are realized in the Ring. The real failure is the following: attempts of the strict justification of its existence in the work [8] are incorrect. The writing of variational equations as such presuppose the differentiability of the equation (8) relative to the initial data. However, it was proved in the theorem 2 that the equation (8) can not have stationary, differentiable solutions. Hence, all the written out oscillating modes were obtained from internally inconsistent premises.

This failure is typical for many works. Using calculations obtained from two-dimensional model and meeting obstacles, one gets out of difficulty by appealing to three-dimensional real situation. This position is contradictory. In contrast of this, we explicitly declare that the two-dimensional model does not have smooth solutions. The exit is to consider only three-dimensional model, or to seek non-smooth (may be a fractal) solutions of the two-dimensional model

Let us complicate the model by adding the impact of the Saturn moons on particles of the Ring. Let $Z_i(t),\ i = 1...l$ be Cartesian coordinates of moons at a moment t and ν_i being its masses. The resultant gravitational force acting on a point $X \in Q$ equals

$$\Phi(t, X) = \mathcal{F}(X) + \sum_{i=1}^{l} \frac{\nu_i}{|Z_i(t) - X|^2}.$$

The Boltzmann equation takes the form

$$\begin{array}{c} \frac{\partial f_\mu(t,X,V)}{\partial t} + \sum_{i=1}^{2} \frac{\partial f_\mu(t,X,V)}{\partial x_i} v_i + \sum_{i=1}^{2} \frac{\partial f_\mu(t,X,V)}{\partial v_i} \Phi_i(t, X) = \\ \frac{1}{\lambda} \int_0^\infty s(m) dm \int K(\sigma)[\tilde{f}_\mu \tilde{f}_m - f_\mu f_m] d\omega_\mu d\sigma. \end{array} \tag{16}$$

Our point of view is the following: The genuine reason for stratification of the Ring consists in the fractal solution of the Boltzmann equation connected with the piece-wise smooth Hamiltonian system that is the limit-system for that of (16). The fractal solution is the optimal solution to the affine in control extremal problem of minimization of the functional of the action function. The part of the control play gravitational forces from the Moons of Saturn $\frac{\nu_i}{|Z_i(t) - X|^2}$ *. The sum of these forces takes its values in the polyhedron with vertices in positions of the Moons* $Z_i(t)$ *.*

Let us explain. Long ago it has been established that the basis for the lows of movement are variational principles. This concept allows to introduce the theory of the Saturn Ring into the framework of the optimal control theory and of the variational problems.

Particles of the Ring moves, in response to the gravitational force of Saturn, mainly by circles, creating the structure of the Ring. Hence, in view of stationarity of the movement, it shows the phase portrait of trajectories emanating practically from all initial points of the Ring. By other words, we see the projection on the configuration space of the optimal synthesis for the minimization problem of the action function.

After Poincaré, the exploration of the structure of trajectories of ordinary differential equations (ODE) with smooth right hand size is based on the analysis of basic singularities

of model system: its singular points, limit cycles, attractors and so on. After that, one proves that the phase portrait, being perturbed, remains topologically equivalent to that of the initial model system. Hamiltonian systems with discontinues right hand side arising from Pontryagin Maximum Principle have specific structure (so called tangential jumps). In this case the specific phenomenon of general position takes place: the infinite number of switchings on finite intervals of trajectories (I Kupka [12], M.I.Zelikin and V.F.Borisov [24]). In [24] the theorem on fiber bundle was proved. Roughly speaking, it claims:

Perturbations of Fuller's problem by members of higher order relative to the scale group and by adding rectifiable auxiliary variables do not change the topological structure of Fuller problem synthesis (in particular, the infinite number of switchings on finite intervals of trajectories). The neighborhood of singular extremals of second order is fibred by manifolds having perturbed Fuller's problem synthesis.

Hence, it was proved that the synthesis of the Fuller problem is a model for Pontryagin-type Hamiltonian systems with one-dimensional control. The generalizations of this construction for problems with one-dimensional control in infinite-dimensional spaces were obtained in [25]. In case of multi-dimensional control it was found modes, where the optimal control performs in a finite time the countable number of revolutions, and also the modes where the optimal control accomplishs in a finite time the full circuit along the everywhere dense winding of a torus [26].

This approach was extended to problems with the affine multidimensional control when control variables take values in a polyhedron. It was discovered a new general phenomenon for Pontryagin-type Hamiltonian systems with discontinues right hand side — the stochastic dynamics — the full circuit of Cantor-like non-wandering points which is realized in a finite time [27].

Naturally, one cannot apply this theorem directly to the behavior of particles of the Saturn Ring.

At the first glance the theorem on piece-wise smooth Hamiltonians is inapplicable to our case. Hamiltonian of the many body problem is smooth. However, if one considers the action of several bodies on a test particle, then in the process of moving arise situations when the Hamiltonian appears as close as desired to a piece-wise smooth one. Though linear combinations of gravitational forces from Moons lies strictly inside of the polyhedron with vertices in positions of Moons and the set of controls is an open polyhedron, however, optimal trajectories for the problem with the open polyhedron can be approximated by those of closed ones. But the corresponding passage to the limit for the entire optimal synthesis demands justifications.

Besides, the theorem of Zelikin-Lokutsievskiy-Hildebrand relates to the ordinary differential equations. The Boltzmann equation is integro-differential and infinite-dimensional. Nevertheless, the more reason is to expect the fractal structure of its solutions in projection of an infinite-dimensional picture on the finite-dimensional configuration space.

The key idea is the following: On particles act external (relative to the Ring and Saturn) forces – that from numerous Moons of Saturn. Although one can not dispose its positions, nevertheless, they can be regarded as control, moreover, as the optimal control. Indeed, the Moons together with Saturn and its Ring as a whole, behave as if it were their aim to minimize the functional of action. In this process the control being a linear combination of forces applied at the finite number of points $Z_i(t)$ (corresponding to disposition of Moons) changes in a polyhedron.

Conjecture 1. *Kinetic Boltzmann equation related to the system (16) has a fractal solution.*

To use an abstract mathematical theorem in a real physical situation demand caution. The exact fractal solution may be observed if the asteroids constituent the Ring were ideal points. But as far as they have a finite size, although insignificantly small in comparison with the magnitude of the Ring itself, one observes only the approximative picture: the tremendous number inserted one into another isolated ringlets consisted of particles of finite size. This matter is too rough to realize theoretical Cantor-like set of orbits. The situation call to mind that one which took place while the Lorenz attractor was discovered: the better were optical instruments the bigger number of ringlets one could see. Particles of the Ring do its best (as far as it would be possible for particles of centimetre-sized diameters) to reproduce the exact fractal solution that minimizes the functional of action. Different oscillating processes in the Ring are simply vibrations around the faithful fractal solution.

The influence of the electromagnetic factors on the Ring structure is expected explorations and explanations. This influence must be considerable. The electric impulses, that were accepted by interplanetary apparatus Voyager with the source being localized in the region of the Ring, were 100000 times as large as the most powerful lightnings in the Earth atmosphere. The radial strips, so called "spokes", observed on the Ring, rotate as a distinct units, retaining its straight form, independently of the differential rotation of the whole ambient "wheel", with the speed of rotation of the magnetic field of Saturn. The spacecraft Voyager detected the flow of very thin charged particles (with the size of approximately 10^{-8} m) having the speed close to 100 km/sek. It is natural to suppose that, due to the shear friction, the differential rotation of particles must induce the ionization of the mass of very thin dust and the accumulation of very big charges in view of the vacuum isolation. As far as the Boltzmann equation is applicable both to the kinetic effects and to the behaviour of plasma, it seems to be an appropriate technique for these explorations.

5 Concluding remarks

Let us remark, that all discovered Rings of planets (around Jupiter, Saturn, Uranus, and Neptune) consist of the set of inserted one into another ringlets. Besides, the Solar System itself has a Ring — the Asteroid Belt situated between orbits of Mars and Jupiter.

Our experience in exploration of the topological structure of phase portraits and the proved fact about the fractal structure of optimal syntheses in general situation, gives grounds to formulate the following conjecture:

Conjecture 2. *The Asteroid Belt of the Solar System approximates the exact fractal solution of the corresponding variational problem and consists of the very big number isolated, inserted one into another ringlets. The part of shepherds play planets of Solar System — Mars, Jupiter, Saturn, Uranus...*

Surely, all the planet of the Solar System define corresponded mobile polyhedron; its gravitational forces realize control of the Asteroid Belt. These forces should be considered not as external impacts that call resonances but as an intrinsic forces of the full system minimizing the action function. That must be the cause of fractal structure of the Asteroid Belt.

Gaps in the Asteroid Belt have been discovered. They are called the Kirkwood gaps. Periods of its rotation until now were associated with that of Jupiter. Admittedly, Mars has significantly less mass than Jupiter but in the course of its rotation Mars practically intersects the Asteroid Belt. In addition the impact of other planet should be worth to take into account.

We know that the big planets of the Solar System, and the Solar System itself have Rings. So the right statement of the question is not to ask: why planets have rings, but to ask: why the planets of the Earth group (Mercury, Venus, Earth, and Mars) are devoid of Rings. The standard preliminary conjecture is that they are too near to Sun, where the density of the interplanetary dust is too poor to complete its reduction due to accretion on planets. May be rings do not sustain against the pressure of the Sun wind, or the Sun light?

But in accordance with the our conception the cause can include in the dimension of control. The theorem of Zelikin-Lokutsievskiy-Hildebrand [27], [28] demands that the dimension of control must be greater than 1. Mars has only two satellites Fobos and Deimos, and the control is one-dimensional. Earth has only one satellite. Venus and Mercury do not have satellites. To the contrary, Jupiter, Saturn, Uranus, and Neptune (all of them have Rings) have many satellites. Our point of view is the following: satellites do not only stratify rings. The creation and the maintenance of the existence of Rings are connected with the presence of several big satellites. They organize streams of meteorites, gathering its into a Ring, and maintain the existence of the Ring by giving to it the fractal structure.

We insist on the following proposition:

Existence of Rings and its stratification on ringlets is not the exclusive but the regular phenomenon that is inherent in sufficiently large planet systems

Let us formulate the following program that seems to us very important and very difficult as for the theory of kinetic Boltzmann equation itself, so also for its numerous applications in kinetic theory of gases, in cosmogony, in dynamics of plasma and in other domains.

Program 1. *Investigation of fractal solutions to kinetic Boltzmann equation.*

The first step to realize this program would be to explore the above-mentioned simplest case and to prove of the conjecture 1.

We only have to say that on the way of this proof stands considerable mathematical difficulties. Questions of the existence, of the smoothness, and of the extendability of solutions to the Boltzmann equation evoke serious discussions among professionals [9],[3] [19] [23]. Since the proof of the existence of the fractal solution to ordinary differential equations appears very difficult (the exact and thorough proof takes about hundred pages), the analog of the theorem for the integro-differential equation must be much more difficult.

But even this would not solve the problem because the Boltzmann equation itself was derived under the assumption that the density function is smooth.

Both Maxwell and Poincaré refused (not without reason) to make final conclusions about the stability of Saturn Ring, although namely this question was the subject of their searchings. Moreover, Maxwell was led to conclude that the Ring will evolute and eventually disintegrates in the foreseeable future. After receiving from Struve observatory some not so high-reliable information about changes in the size of the Saturn Ring, Maxwell wrote:

"It will be worth while to investigate more carefully whether Saturn's Rings are permanent or transitionary elements of the Solar System, and whether in that part of the heavens we see celestial immutability or terrestrial corruption and generation, and the old order giving place to new before our own eyes".

Bibliography

[1] P.Appell. "Trait/'e de M/'echanique Rationell". Paris, Gauthier-Villard. (1911)

[2] L.Boltzmann. "Vorlesungen über die gasen theorie". (1898)

[3] T.Carleman. "Problemés mathematiques dans la theorie cinematiques des gaz". Almvist and Wiksells, Upsala (1957)

[4] S.Chapman and T.J.Cowling. "The mathematical theory of non-uniform gases". Cambridg (1939)

[5] M.Couette. Ann.Chim.Phys. 21 433 (1890)

[6] D.Gibbs. "Thermodynamische Studien".(1892)

[7] H.Grad. Commun.Pure Appl.Math. 2.(1949).p.311

[8] N.N.Gorkavyj, A.M.Friedman. "The Physics of planetary rings". Russian Phys.Surveys.V.160(2)(1990),pp.169-237

[9] R.L.Dobrushin. "Vlasov's equation". Func.Anal.and its Appl. 13 (1979) p.115-123

[10] G.Kirchhoff. "Vorlesungen über Wärmetheorie".

[11] Kowalewski (M-m) "Zusätze und Bemerkungen zu Lahlace's Untersuchung über die Gestalt der Saturnsringe". Astronomische Nachrichten, t.CXI, 1885,№ 2643.

[12] I. Kupka. "Fuller's phenomena", Perspectives in Control Theory. (Sielpia, 1988) Progr. Systems Control Theory. Birkhäuser, Boston. (1990), pp.129-142

[13] P.S.Laplace. "Memoir sur la theory del l'anneau de Saturn". Mem. de l'Acad des Sci. – Mechanique celeste. (1787)

[14] J.F.Lutsko. "Chapman-Enskog expansion about nonequilibrium states with application to the sheared granular fluid". Physical Review E 73. 021302 (2006)

[15] J.C.Maxwell. "On the stability of the motion of Saturn's Rings". in: The Scientific Papers of James Clerk Maxwell. Dover. N-Y. (1890)

[16] J.C.Maxwell. "On the dynamical theory of gases". Phil.Trans.of Roy.Soc.I, (1879); Scient.Pap.II

[17] H.Poincaré. "Sur la Figure de la terre".Oeuvres V.VIII.(1889)

[18] H.Poincaré. "Note sur la stabilité de l'anneau de Saturne". Bull.Astron. (1885) 2. p.507-508

[19] A.Ja.Povzner. Mat.Sbornik. 58, 65 (1962)

[20] Lord Rayleigh. Proc.R.Soc.London. A 93 148 (1917)

[21] D.A.Shalybkov. "Hidrodynamic and hydromagnetic stability of the Couette flow". Russian Phys.Surveys. 179(9) (2009.pp.971-993

[22] A.Sommerfeld. "Thermodynamik und Statistik". Wiesdaden.(1952)

[23] S.Ukai, T.Yang. "Mathematical Theory of Boltzmann Equation". University of Hong Kong.(2006)

[24] M.I.Zelikin, V.F.Borisov. "Theory of Chattering Control with applications to Astronautics, Robotics,Economics, and Engineering". Birkhauser, Boston, (1994)

[25] M.I.Zelikin, V.F.Borisov, L.A.Manita "Optimal Synthesis in an Infinite-Dimensional Space". Proc. Steklov Math. Inst. Vol.271.(2010) pp.40-58

[26] M.I.Zelikin, L.V.Lokutsitvskiy, R.Hildebrand. "Geometry of Neighborhood of Singular trajectories in Problems with Multidimensional Control". Proc.Steklov Math.Inst. V.277, pp.67-83. (2012)

[27] M.I.Zelikin, L.V.Lokutsitvskiy, R.Hildebrand. "Generic Fractal Structure of Finite Parts of Trajectories of Piecewise Smooth Hamiltonian Systems". Russian Journal Math.Phys. Vol.20, №4, (2012), pp.45-52

[28] M.I.Zelikin, L.V.Lokutsitvskiy, R.Hildebrand. "Typicalness of Chaotic Fractal Behaviour of Integral Vortexes in Hamiltonian Systems with Discontinuous Right Hand Side". arXiv:1506.02320